On univariate Chebyshev approximation

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Abstract

Chebyshev polynomials of the first kind are a popular choice for basis polynomials in function approximation due to their closeness to the Fourier cosine series, their minimization of the worst-case interpolation error among all monic polynomials on [-1,1], and the unique representation and uniform convergence of the Chebyshev series expansion to any function on $C([-1,1])$. Below, I show equivalent formulations derived from the fundamental recurrence defining first-kind Chebyshev polynomials 1.1, then I collect some important properties for approximations with these polynomials 1.2. Lastly, I discuss the Chebyshev series 1.3 and interpolation 1.4.

Notation Unless denoted otherwise, $||\cdot|| = ||\cdot||_2$. $\mathbb{F}$ denotes the field of numbers; the results hold for $\mathbb{R}$ and $\mathbb{C}$. Regular lowercase letters, e.g., x, denote scalars. Bolded lowercase letters, e.g., $\mathbf{x}$, denote vectors. Uppercase regular letters, e.g., X, denote matrices. Caligraphic uppercase letters, e.g., $\mathcal{X}$, denote tensors.

1 Univariate Chebyshev Approximation

1.1 Basic Properties of Chebyshev polynomials

Lemma 1.1. *(First-kind Chebyshev polynomials [[1], Ch. 1]) The following are equivalent:*

1.
$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \end{aligned} \tag{1}$$

2.
$$T_n(\cos\theta) = \cos(n\theta) \iff T_n(x) = \cos(n\cos^{-1}(x)) \tag{2}$$

3.
$$T_n(x) = \frac{1}{2}[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n] \tag{3}$$

Proof: Starting from 1, 2 is verified by induction using de Moivre's formula and the identity $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)$. 3 is derived from 2 using Euler's formula and the fact that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$ for $|x| \leq 1$. $\square$

Lemma 1.2. *(Properties of first-kind Chebyshev polynomials [[1], Ch. 2-4])*

1. *$T_n(x)$ has n roots on [-1,1] at $x_k = \cos(\frac{(2k+1)\pi}{2n})$ for $k = 0, ..., n-1$.*

2. *$\max |T_n(x)| = 1$ on [-1,1] at $x_k = \cos(\frac{k\pi}{n})$ for $k = 0, ..., n$.*

3. *$\min_{P\in\mathcal{P}_n(\mathbb{F})} ||P||_{\infty,[-1,1]} = \frac{T_n}{2^{n-1}}$.*

4. *For $\mu(x) = \frac{1}{\sqrt{1-x^2}}$, $\langle T_i, T_j\rangle_\mu = \begin{cases} \pi, & i=j=0 \\ \frac{\pi}{2}, & i=j\neq 0 \\ 0, & i\neq j \end{cases}$*

where $\mathcal{P}_n(\mathbb{F})$ denotes the set of monic n-degree polynomials. Proof:

1. Since $\cos((2k+1)\frac{\pi}{2}) = 0$, $\cos(n\cos^{-1}(x)) = 0 \implies x_k = \cos(\frac{(2k+1)\pi}{2n})$ for $k = 0, ..., n-1$.
2. $|\cos(x)| \leq 1$ and $\cos(k\pi) = \pm 1 \implies \cos(n\cos^{-1}(x)) = \pm 1$ when $x_k = \cos(\frac{k\pi}{n})$ for $k = 0, ..., n$.
3. Assume $P^* \in \mathcal{P}_n(\mathbb{F})$ has the minimal max absolute value of all n-degree monic polynomials $\mathcal{P}_n(\mathbb{F})$ on $[-1, 1]$. Let $f(x) = \frac{T_n(x)}{2^{n-1}} - P^*(x)$, which is an $(n-1)$-degree polynomial. This implies $|\frac{T_n(x)}{2^{n-1}} - P^*(x)| > 0$ at the extremal points $x_k = \cos(\frac{k\pi}{n})$, i.e., P^* oscillates below $\frac{T_n(x)}{2^{n-1}}$ at all extremal points on the interval. $T_n(x_k) = -T_n(x_{k+1})$ implies $f(x)$ oscillates and since there are $n+1$ extremal points, there must be n roots by the intermediate value theorem, which is a contradiction. By the fundamental theorem of algebra, $f(x)$ can have at most $n-1$ roots. Therefore, no other $P \in \mathcal{P}_n(\mathbb{F})$ can improve on $\frac{T_n(x)}{2^{n-1}}$ without adding roots in [-1,1] (aka adding degrees).
4. Using 2 and the identity $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$, one can show $\langle T_i, T_j\rangle_\mu = \int_{-1}^{1} \frac{1}{2}(\cos((i-j)\cos^{-1}(x)) + \cos((i+j)\cos^{-1}(x)))\mu(x)dx$. From that integral, the case for $i = j = 0$ is immediate. For $i = j \neq 0$, using the fact that $\frac{d\cos^{-1}}{dx} = \frac{1}{\sqrt{1-x^2}}$, we see that $\int_{-1}^{1} \cos((i+j)\cos^{-1}(x)))\mu(x)dx = 0$, leaving only $\int_{-1}^{1} \frac{1}{2}\mu(x)dx = \frac{\pi}{2}$. For $i \neq j$, both integral terms evaluate to 0, as in the previous case.

□

Theorem 1.3. *(Chebyshev series [[1], Ch. 5; [2], Theorem 3.1]) Lipschitz continuous $f : [-1, 1] \to \mathbb{F}$ is uniquely represented as a Chebyshev series:*

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \tag{4}$$

which is absolutely and uniformly convergent, with coefficients:

$$a_k = \frac{\langle f, T_k\rangle_\mu}{||T_k||_\mu^2} = \begin{cases} \frac{1}{\pi}\langle f, T_k\rangle_\mu, & k = 0 \\ \frac{2}{\pi}\langle f, T_k\rangle_\mu, & k > 0 \end{cases} \tag{5}$$

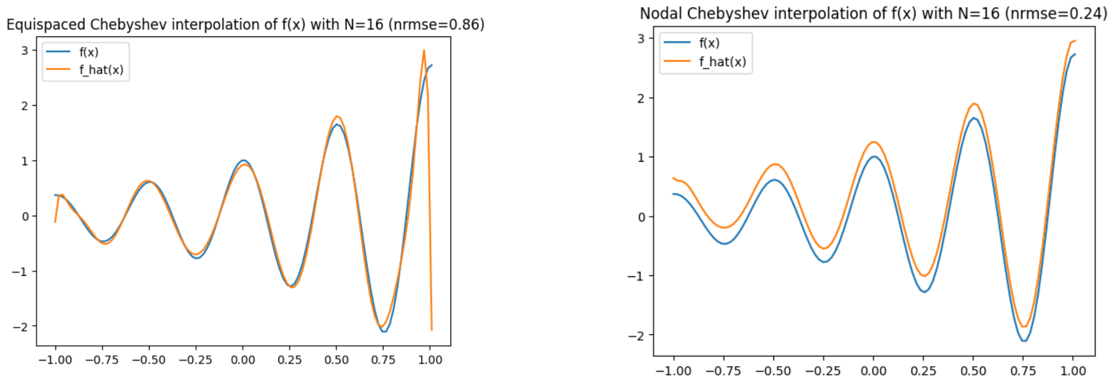
Proof: For 4, by the Stone-Weierstrass theorem [[3], Theorem 7.26], we know that a sequence of polynomials on [-1,1] exists that converges uniformly to any continuous complex function on [-1,1]. Since $\{T_k\}_{k=0}^{\infty}$ form an orthogonal basis in $L^2_\mu([-1, 1])$, any function in $L^2_\mu([-1, 1])$ can be represented as a linear combination of the basis functions by the definition of a Hilbert space; therefore, any such function can be represented by the Chebyshev basis. For absolute and uniformly convergent, the coefficients are bounded and decay exponentially [[2], Theorem 8.1]. Hence, the series is absolutely convergent and, since $\frac{dT_n}{dx} = nU_{n-1}$ where $U_n(x) = \frac{\sin((n+1)\cos^{-1}(x))}{\sqrt{1-x^2}}$ is the second-kind Chebyshev polynomial, the series of derivatives is convergent, implying that the series is uniformly convergent on [-1,1] by [[3], Theorem 7.17]. To obtain the optimal coefficients 5, we minimize $F(\mathbf{a}) = ||f - \sum_{i=0}^{\infty} a_i T_i||_\mu^2$. Note that:

$$\begin{aligned}
||f - \sum_{i=0}^{\infty} a_i T_i||_\mu^2 &= ||f||_\mu^2 - \sum_i^{\infty} a_i \langle T_i, f\rangle_\mu - \sum_i^{\infty} a_i^* \langle f, T_i\rangle_\mu + \sum_{i,j}^{\infty} a_i a_j^* \langle T_i, T_j\rangle_\mu \\
\implies \frac{\partial F}{\partial a_i^*} &= \frac{1}{2}[-\langle f, T_i\rangle_\mu + a_i ||T_i||_\mu^2] = 0 \\
\implies a_i &= \frac{\langle f, T_i\rangle_\mu}{||T_i||_\mu^2}
\end{aligned} \tag{6}$$

where the Wirtinger derivative is taken w.r.t. the conjugate of a_i and set to 0 to satisfy the Cauchy-Riemann conditions [4]. □

1.2 Chebyshev Interpolation

Although the analytic solution for a coefficient holds for the truncated series, the challenge is efficiently approximating the inner product $\langle f, T_i\rangle_\mu$ on a finite grid and window. A first choice would be to take densely-sampled equispaced points. However, it's well-known that equispaced sampling is often a suboptimal



(a) (b)

Figure 1: Comparison of equispaced (a) and nodal (b) Chebyshev interpolation of $f(x) = e^x \cos(4\pi x)$. Both fit a 16-degree polynomial. (a) uses the trapezoidal rule to approximate the inner product from 100 samples. At the edges, the interpolation destabilizes.

scheme for finite grids on finite windows, leading to large errors at the boundary points [[1], Theorem 6.1]. For uniform convergence, a better choice is to approximate the coefficients at the $n+1$ nodes of T_{n+1}. This is also more efficient, especially since $n+1$ tends to be relatively small leading to fewer function evaluations and no need for trickery like the trapezoidal rule or complicated quadrature methods.

Theorem 1.4. *(Chebyshev interpolant coefficients [[1], Theorem 6.7])*

For the Chebyshev nodes of T_{n+1},

$$x_k = \cos\left(\frac{(k-\frac{1}{2})\pi}{n+1}\right) \textit{ for } k = 1, ..., n+1 \tag{7}$$

the coefficients are given by:

$$c_i = \frac{2}{n+1}\sum_{k=1}^{n+1} f(x_k)T_i(x_k) \tag{8}$$

Proof: Using the fact:

$$\sum_{k=1}^{n+1} T_i(x_k)T_j(x_k) = \begin{cases} 0, & i \neq j \\ n+1, & i = j = 0 \\ \frac{n+1}{2}, & 0 < i = j \leq n \end{cases}$$

which results from discrete orthogonality (see [[1], Section 4.6.1] for derivation). Interpolating at the $n+1$ Chebyshev nodes, implies:

$$f(x_k) = {\sum_{i=0}^{n}}' c_i T_i(x_k)$$

where $\sum'$ corresponds to weighting the 0th iterate by $\frac{1}{2}$ and subsequent iterates by 1. Multiplying by $\frac{2}{n+1}T_j(x_k)$ and summing for each k gives:

$$\frac{2}{n+1}\sum_{k=1}^{n+1} f(x_k)T_j(x_k) = {\sum_{i=0}^{n}}' c_i \frac{2}{n+1}\sum_{k=1}^{n+1} T_i(x_k)T_j(x_k) = c_j$$

□

1 contrasts between equispaced 1a and nodal 1b interpolation for a simple function.

References

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